Pergamon
www.elsevier.com/locate/jappmathmech

PII: S0021-8928(99)00015-5

J. Appl. Maths Mechs, Vol. 63, No. 1, pp. 101-107, 1999

0021-8928/99/$—see front matter

IMPRESSION WITH ADHESION OF A PUNCH INTO AN ELASTIC HALF-PLANE UNDER A TANGENTIAL LOAD†

I. A. SOLDATENKOV

Moscow

(*Received* 17 *April* 1997)

The approach previously described [1] for solving the contact problem with adhesion is extended to the case of a punch of arbitrary shape acted upon by a tangential load. The contact problem with adhesion for a symmetrical punch with a contact area that increases as a result of the loading was considered in [2-4] using an incremental approach. A solution of this problem was also given in [1] based on inversion, using the method described in [5], of the system of singular integral equations for the contact stresses, and an analysis of the additional conditions which ensure that this inversion is correct.

1. FORMULATION OF THE PROBLEM

Suppose a rigid convex punch is indented into an elastic half-plane (see Fig. 1) by the action of a monotonically increasing normal load P_2 and a tangential load P_1, related to P_2 by a certain relationship

$$P_1 = N(P_2), \quad P_2 > 0 \tag{1.1}$$

When the punch is impressed, adhesion occurs between the contacting bodies, i.e. the points of the half-plane arriving in contact with the punch do not experience any additional displacements with respect to it.

We connect a system of coordinates with the punch, arranging its origin to be at the point of initial contact between the punch and the half-plane (see Fig. 1).

We will assume that the dimensions $a < 0$, $b > 0$ of the contact area increase monotonically as the impressing of the punch progresses. Further, we will characterize by the quantity a (together with the quantity P_2) the degree of impression of the punch, where, in particular, the dimension b will be a certain function of a.

The problem consists of finding the contact stresses $q_1 = \tau_{xy}|_{y=0}$, $q_2 = -\sigma_y|_{y=0}$ for arbitrary a, and also the relationship $b(a)$. The boundary conditions corresponding to the problem in question have the form

$$u(x,a) = \varphi(x) + C_x, \quad v(x,a) = g(x + \mu\varphi(x)) + C_y, \quad x \in [-a,b] \tag{1.2}$$

$$q_1(x,a) = q_2(x,a) = 0, \quad x \,\overline{\in}\, [-a,b]$$

where u and v are the tangential and normal displacements of the boundary of the half-plane in the system x, y, $\varphi(x)$ is a certain function, to be determined, $y = g(x)$ is the equation of the shape of the punch, $\mu = 0, 1$, and the constants C_x and C_y are analogues of the rigid displacement of the punch along the x and y axes, the specific values of which have no effect on the solution of the contact problem, and, moreover, they can be set equal to zero in the chosen system of coordinates.

The presence of the term $\varphi(x)$ in the argument of the function g in (1.2) when $\mu = 1$ corresponds to the more accurate formulation of the contact problem. Namely, it takes into account the fact that, as the punch is impressed, the point x on the boundary of the half-plane is displaced tangentially with respect to it and hence arrives in contact with a point on the punch which has the coordinate $x + \varphi(x)$ [6]. When there is no $\varphi(x)$ in the second equation of boundary conditions (1.2) ($\mu = 0$), the latter will have the classical form [4].

We will impose the following limitations on the shape of the punch and on the unknown function $\varphi(x)$. Namely, we will consider the point $x = 0$ as a node and assume that

$$G(x) \equiv g'(x) \in H_0, \quad \varphi'(x) \in H_0 \tag{1.3}$$

†*Prikl. Mat. Mekh.* Vol. 63, No. 1, pp. 111-118, 1999.

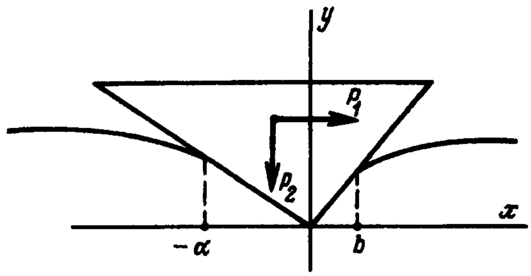

Fig. 1.

We recall that the notation $f(x) \in H_0$ for a node at the point $x = 0$ denotes [5] that, for arbitrary positive d_1 and d_2, the function $f(x)$ belongs to the Hölder class on $[-d_1, 0]$ and $[0, d_2]$, provided that, for the first segment, we take as $f(0)$ the limit of $f(x)$ as $x \to 0 - 0$, and for the second segment the limit of $f(0)$ as $x \to 0 + 0$. Here and henceforth the prime denotes the derivative of a function with respect to the first argument.

We will note some additional properties of the function $\varphi(x)$, the value of which, when $x \in [-a, b]$, is identical with the tangential displacement of the point x of the contact area. First of all, by virtue of the choice of the system of coordinates xy and the conditions of adhesion between the contacting solids, we have

$$\varphi(0) = 0 \tag{1.4}$$

Further, if $x_1 < x_2$, when the half-plane is deformed a point on its boundary with coordinate x_1 will always be situated to the left of the point x_2. This indicates that the sum $x + \varphi(x)$ must be a strictly increasing function of x, which is ensured by the inequality

$$1 + \varphi'(x) > 0 \tag{1.5}$$

As regards the functions $q_1(x, a)$ and $q_2(x, a)$ we will assume that, for each a, they are bounded at the ends—a, b of the contact area and belong to the class H^* in $[-a, b]$, i.e. in the notation used previously in [5] we will assume that

$$q_k(x, a) \in h_2[-a, b], \quad k = 1, 2 \tag{1.6}$$

Note that relation (1.6) allows of the presence in the functions $q_{1,2}(x, a)$ of integrable singularities at the point $x = 0$.

The relation between the boundary stresses and the displacements u and υ, in the linear theory of elasticity, is given by two equalities [7]

$$mu'(x,a) = -\pi\chi q_2(x,a) + \int_{-\infty}^{\infty} \frac{q_1(\xi,a)}{\xi - x} d\xi, \quad mv'(x,a) = -\pi\chi q_1(x,a) - \int_{-\infty}^{\infty} \frac{q_2(\xi,a)}{\xi - x} d\xi \tag{1.7}$$

$$m = \pi E[2(1-\nu^2)]^{-1}, \quad \chi = (1-2\nu)[2(1-\nu)]^{-1}$$

where E is Young's modulus and ν is Poisson's ratio $\nu \in [0, 1/2)$.

We will put $x \in [-a, b]$ in (1.7) and replace the function $u(x, a)$, $\upsilon(x, a)$ in them by the right-hand sides of the corresponding boundary conditions (1.2). We then multiply the first equation of (1.7) by the square root of −1 and add it to the second. We obtain the following equation

$$-\pi\chi q(x,a) + i\int_{-a}^{b} \frac{q(\xi,a)}{\xi - x} d\xi = f(x) \tag{1.8}$$

$$\begin{gathered} q(x,a) = q_1(x,a) + iq_2(x,a), \quad f(x) = \psi_2(x) + i\psi_1(x) \\ \psi_1(x) = m\varphi'(x), \quad \psi_2(x) = mG(x + \mu\varphi(x))(1 + \mu\varphi'(x)) \end{gathered} \tag{1.9}$$

From the definitions of the functions $f(x)$ and $q(x, a)$ and relations (1.3) and (1.6) it follows directly that $f(x) \in H_0$, $q(x, a) \in h_2[-a, b]$.

2. INVERSION OF THE EQUATIONS FOR THE BOUNDARY STRESSES. FUNDAMENTAL RELATIONS

According to the results obtained previously [5], the solution $q(x, a)$ of Eq. (1.8) from the class $h_2[-a, b]$ for a specified a and b and $f(x) \in H_0$ exists if and only if

$$\int_{-a}^{b} \frac{f(\xi)d\xi}{Z_1(\xi,a)} = 0 \tag{2.1}$$

$$Z_1(x,a) = \sqrt{(a+x)(b-x)}\left(\frac{b-x}{a+x}\right)^{i\tau/2}, \quad \tau = \frac{1}{\pi}\ln\frac{1+\chi}{1-\chi} \equiv \frac{1}{\pi}\ln(3-4\nu)$$

When condition (2.1) is satisfied, we have the following formula for inverting Eq. (1.8)

$$q(x,a) = \chi\theta f(x) - \frac{\theta}{\pi i} Z_1(x,a)\int_{-a}^{b} \frac{f(\xi)d\xi}{Z_1(\xi,a)(\xi-x)}, \quad x \in [-a,b]$$

$$\theta = [\pi(1-\chi^2)]^{-1} \tag{2.2}$$

Note that the limit values of the function $q(x, a)$ from (2.2) as $x \to -a + 0$ and $x \to b - 0$ are equal to zero. This follows directly from the well-known theorem on the behavior of the Cauchy-type integral, present in (2.2), close to the ends of the line of integration [5].

We will transform Eqs (2.1) and (2.2). We substitute into (2.1) the expression for the function $f(x)$ in terms of the functions $\psi_1(x)$ and $\psi_2(x)$, rather than the function itself, which, in turn, are related to the functions $\varphi(x)$ and $g(x)$ by Eqs (1.9). Separating the real and imaginary parts in the expression obtained, we arrive at the following two equations

$$\int_{-a}^{b}\left[\varphi'(x)\begin{Bmatrix}-\sin\alpha(x,a)\\ \cos\alpha(x,a)\end{Bmatrix} + G(x+\mu\varphi(x))(1+\mu\varphi'(x))\begin{Bmatrix}\cos\alpha(x,a)\\ \sin\alpha(x,a)\end{Bmatrix}\right]\frac{dx}{\sqrt{(a+x)(b-x)}} = 0 \tag{2.3}$$

$$\alpha(x,a) = \frac{\tau}{2}\ln\left|\frac{a+x}{b-x}\right|$$

Making a similar substitution into (2.2), we will have

$$\begin{Bmatrix}q_1(x,a)\\ q_2(x,a)\end{Bmatrix} = \theta\left[\chi\begin{Bmatrix}\psi_2(x)\\ \psi_1(x)\end{Bmatrix} - \frac{1}{\pi}\sqrt{(a+x)(b-x)}\left(i_1(x,a)\begin{Bmatrix}\sin\alpha(x,a)\\ \cos\alpha(x,a)\end{Bmatrix} + \right.\right.$$

$$\left.\left. + j_1(x,a)\begin{Bmatrix}\cos\alpha(x,a)\\ -\sin\alpha(x,a)\end{Bmatrix} + i_2(x,a)\begin{Bmatrix}\cos\alpha(x,a)\\ -\sin\alpha(x,a)\end{Bmatrix} - j_2(x,a)\begin{Bmatrix}\sin\alpha(x,a)\\ \cos\alpha(x,a)\end{Bmatrix}\right)\right] \tag{2.4}$$

$$\begin{Bmatrix}i_k(x,a)\\ j_k(x,a)\end{Bmatrix} = \int_{-a}^{b}\begin{Bmatrix}\sin\alpha(\xi,a)\\ \cos\alpha(\xi,a)\end{Bmatrix}\frac{\psi_k(\xi)d\xi}{\sqrt{(a+\xi)(b-\xi)}(\xi-x)}, \quad x \in [-a,b]$$

We now return to Eqs (2.3) and note that the quantity b occurring in them is a certain function of a, while these equations themselves must be satisfied for any a. Hence, Eqs (2.3) can be regarded as Volterra-type non-linear integral equations of the first kind in terms of the unknown functions $\varphi(x)$ and $b(a)$.

Relation (1.1) gives one further equation for determining $\varphi(x)$ and $b(a)$. In fact, from the equilibrium condition for the punch we have

$$P_1 = \int_{-a}^{b} [q_1(x,a)\cos\omega(x) + q_2(x,a)\sin\omega(x)]dx$$

$$P_2 = \int_{-a}^{b} [-q_1(x,a)\sin\omega(x) + q_2(x,a)\cos\omega(x)]dx \tag{2.5}$$

where $\omega(x) = \operatorname{arctg} g'(x)$. Substituting (2.5) into (1.1) and taking expressions (2.4) for $q_{1,2}(x, a)$ into account, we arrive at the additional equation for $\varphi(x)$ and $b(a)$.

Hence, the three equations, namely, the two equations (2.3) and Eq. (1.1), together with expressions (2.4) and (2.5), form a system of equations in terms of the unknown functions $\varphi(x)$ and $b(a)$, after finding which, expressions (2.4) determine the stresses $q_{1,2}(x, a)$, thereby solving the problem in question.

Note that the second of equations (2.5) for the known stresses $q_{1,2}(x, a)$ gives the value of the normal load P_2 as a function of the dimension a of the contact region. As was mentioned above, as the punch penetrates the positive load P_2 increases monotonically, and hence we have the inequalities

$$P_2(a) > 0, \quad P_2'(a) \geqslant 0 \tag{2.6}$$

which, together with (1.4) and (1.5) impose additional constraints on the functions $\varphi(x)$ and $b(a)$.

Below we consider the special case of a wedge-shaped punch and the linear relationship (1.1), for which the unknown function $\varphi(x)$ can be found in the class of piecewise-linear functions for a linear form of $b(a)$.

3. A WEDGE-SHAPED PUNCH (THE GENERAL CASE)

Suppose

$$g(x) = \begin{cases} g_1^- x, & x < 0 \\ g_1^+ x, & x \geqslant 0 \end{cases}; \quad \begin{matrix} g_1^\pm = \text{const} \\ g_1^- < 0, \ g_1^+ > 0 \end{matrix} \tag{3.1}$$

Henceforth we will set $|g_1^\pm| < 1$, which is necessary in order to justify the use of Eqs (1.7) of the linear theory of elasticity. We will seek the unknown function $\varphi(x)$ in the class of piecewise-linear functions, namely, taking Eq. (1.4) into account we set

$$\varphi(x) = \begin{cases} \varphi_1^- x, & x < 0 \\ \varphi_1^+ x, & x \geqslant 0 \end{cases} \tag{3.2}$$

Finally, we assume relation (1.1) to be linear

$$P_1 = nP_2, \quad n = \text{const} \tag{3.3}$$

It is easy to establish that expressions (3.1) and (3.2) for the functions $g(x)$ and $\varphi(x)$ satisfy conditions (1.3). We substitute these expressions into (2.3) and (3.3), where, in the last equality, we define the quantities $P_{1,2}$ in terms of $\varphi(x)$ according to the following chain of equalities: (2.5), (12.4) and (1.9). After simple calculations we thereby arrive at the following system of three equations

$$\begin{cases} \alpha_{11}\varphi_1^- + \alpha_{12}\varphi_1^+ = \beta_1 \\ \alpha_{21}\varphi_1^- + \alpha_{22}\varphi_1^+ = \beta_2 \\ \alpha_{31}\varphi_1^- + \alpha_{32}\varphi_1^+ = \beta_3 \end{cases} \tag{3.4}$$

$$\alpha_{11} = (\gamma_0 - \Lambda_1) + \mu g_1^-(\delta_0 - \Lambda_2), \quad \alpha_{12} = -(\gamma_0 - \Lambda_1) + \mu g_1^+(\delta_0 + \Lambda_2)$$

$$\alpha_{21} = (\delta_0 - \Lambda_2) - \mu g_1^-(\gamma_0 - \Lambda_1), \quad \alpha_{22} = (\delta_0 + \Lambda_2) + \mu g_1^+(\gamma_0 - \Lambda_1)$$

$$\alpha_{31} = B_1^- - nB_2^-, \quad \alpha_{32} = B_1^+ - nB_2^+$$

$$\beta_1 = -g_1^-(\delta_0 - \Lambda_2) - g_1^+(\delta_0 + \Lambda_2), \quad \beta_2 = -(g_1^+ - g_1^-)(\gamma_0 - \Lambda_1)$$

$$\beta_3 = -C_1 + nC_2$$

$$B_1^\pm = \pm(A_1^- + \mu g_1^\pm A_2^-)\sin\omega^- \pm (A_1^+ + \mu g_1^\pm A_2^+)\sin\omega^+ \mp$$

$$\mp(A_2^- - \mu g_1^\pm A_1^-)\cos\omega^- \mp (A_2^+ - \mu g_1^\pm A_1^+)\cos\omega^+$$

$$B_2^\pm = \pm(A_2^- - \mu g_1^\pm A_1^-)\sin\omega^- \pm (A_2^+ - \mu g_1^\pm A_1^+)\sin\omega^+ \pm$$

$$\pm(A_1^- + \mu g_1^\pm A_2^-)\cos\omega^- \pm (A_1^+ + \mu g_1^\pm A_2^+)\cos\omega^+$$

$$C_1 = (g_1^+ - g_1^-)(A_2^- \sin\omega^- + A_2^+ \sin\omega^+ + A_1^- \cos\omega^- + A_1^+ \cos\omega^+)$$

$$C_2 = (g_1^+ - g_1^-)(-A_1^- \sin\omega^- - A_1^+ \sin\omega^+ + A_2^- \cos\omega^- + A_2^+ \cos\omega^+)$$

$$A_1^\pm = \pm[(\pm\delta_0 + \Lambda_2)\sin\tau\rho + (\gamma_0 - \Lambda_1)\cos\tau\rho]/\operatorname{ch}\rho$$

$$A_2^{\pm} = [\mp(\gamma_0 - \Lambda_1)\sin\tau\rho + (\delta_0 \pm \Lambda_2)\cos\tau\rho]/\mathrm{ch}\,\rho$$

$$\begin{Bmatrix}\Lambda_1\\ \Lambda_2\end{Bmatrix} = \begin{Bmatrix}\Lambda_1(\rho)\\ \Lambda_2(\rho)\end{Bmatrix} \equiv \int_0^{\delta} \begin{Bmatrix}\sin\tau X\\ \cos\tau X\end{Bmatrix}\frac{dX}{\mathrm{ch}\,X},\quad \rho = \frac{1}{2}\ln\frac{b}{a}$$

$$\gamma_0 = \Lambda_1(\infty),\quad \delta_0 = \Lambda_2(\infty),\quad \omega^{\pm} = \operatorname{arctg}_1^{\pm}$$

We note immediately the following properties of the quantities introduced, which can be established by a simple analysis when $\nu \in [0, 1/2]$

$$0 < \gamma_0 < \delta_0;\ \Lambda_{1,2}(\rho) \neq 0 \quad \text{for } \rho \neq 0 \tag{3.5}$$

Equations (3.4) are linear algebraic equations in the unknown parameters φ_1^-, φ_1^+ of the function $\varphi(x)$. The coefficients of Eqs (3.4) are determined by the constants $g_1^{\pm}$, n, τ and μ, and also by the unknown quantity ρ, which describes the degree of asymmetry of the contact area. Taking this fact into account, we require the function $b(a)$ to be linear, which ensures that the quantity ρ is constant as α increases, and consequently (by virtue of Eqs (3.4)), the constancy of the parameters $\varphi_1^{\mp}$, laid down in (3.2). Hence, we will assume that

$$b(a) = e^{2\rho}a,\quad \rho = \mathrm{const} \tag{3.6}$$

We will denote by A^* the augmented 3×3 matrix of system (3.4), which is formed by adding the column of free terms of system (3.4) to the matrix of the coefficients of its left-hand side. We know [8], that the necessary condition for a solution $\varphi_1^{\mp}$ of system (3.4) to exist for any fixed ρ is the equation

$$\det A^* = 0 \tag{3.7}$$

The left-hand side of (3.7) contains the single unknown quantity ρ and hence this condition can be regarded as the equation for ρ. Using the formula for the expansion of the determinant in (3.7) with respect to the third row of the matrix A^* [8], we obtain the following representation for Eq. (3.7)

$$S_1(\rho) - nS_2(\rho) = 0 \tag{3.8}$$

in which $S_{1,2}(\rho)$ are known functions, defined by the quantities $g_1^{\mp}$ and μ

$$S_k(\rho) = C_k d_0 + B_k^- d_1 + B_k^+ d_2 \tag{3.9}$$

$$\begin{gathered} d_0 = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \\ d_1 = \alpha_{22}\beta_1 - \alpha_{12}\beta_2,\quad d_2 = \alpha_{11}\beta_2 - \alpha_{21}\beta_1 \end{gathered} \tag{3.10}$$

The quantity ρ obtained from Eq. (3.8) by means of Eq. (3.6) defines the linear relationship $b(a)$. Moreover, for known ρ from Eqs (3.4) (one of which is eliminated from consideration since, by virtue of (3.7), it is a consequence of the remaining ones [8]) we obtain the parameters $\varphi_1^{\mp}$ and, consequently, also the function $\varphi(x)$.

After finding $b(a)$ and $\varphi(x)$, the stresses $q_{1,2}(x, a)$ are found by simple substitution of expressions (3.1), (3.2) and (3.6) into (2.4)

$$\begin{gathered} q_1(x,a) = \theta\{\chi(\psi_2(x) - \psi_2^+) + \frac{1}{\pi}[(\psi_2^+ - \psi_2^-)W_1(t) - (\psi_1^+ - \psi_1^-)W_2(t)]\} \\ q_2(x,a) = \theta\{\chi(\psi_1(x) - \psi_1^+) + \frac{1}{\pi}[(\psi_1^+ - \psi_1^-)W_1(t) + (\psi_2^+ - \psi_2^-)W_2(t)]\} \\ \psi_1^{\pm} = m\varphi_1^{\pm},\quad \psi_2^{\pm} = mg_1^{\pm}(1 + \mu\varphi_1^{\pm}),\quad t = \frac{1}{2}\ln\frac{1 + x/a}{1 - x/b} \\ \psi_k(x) = \begin{cases}\psi_k^-, & x < 0\\ \psi_k^+, & x \geqslant 0\end{cases};\quad \begin{Bmatrix}W_1(t)\\ W_2(t)\end{Bmatrix} = \int_t^{\infty}\begin{Bmatrix}\sin\tau X\\ \cos\tau X\end{Bmatrix}\frac{dX}{\mathrm{sh}\,X} \end{gathered} \tag{3.11}$$

The known stresses $q_{1,2}(x, a)$ enable us, using the second equation of (2.5), to obtain $P_2(a)$

$$P_2(a)=\frac{m_0}{2}(1+e^{2\rho})(B_2^-\varphi_1^-+B_2^+\varphi_1^++C_2)a,\quad m_0=\frac{m}{\pi^2(1-\chi^2)} \tag{3.12}$$

Finally, we need to verify whether the expressions obtained for $\varphi(x)$ and $P_2(a)$ satisfy inequalities (1.5) and (2.6), which, by virtue of expressions (3.2) and (3.12) have the form

$$1+\varphi_1^\pm>0,\quad B_2^-\varphi_1^-+B_2^+\varphi_1^++C_2>0 \tag{3.13}$$

If at least one of inequalities (3.13) is violated, expressions (3.2), (3.6) and (3.11), for the values of ρ and $\varphi_1^\pm$ obtained, do not give a solution of the problem of the indentation of a wedge-shaped punch with adhesion into a half-plane. Such a situation indicates that there is no solution of the problem in question in the class of piecewise-linear functions for the tangential boundary displacement $\varphi(x)$ when $b(a)$ is linear.

4. A SYMMETRICAL WEDGE-SHAPED PUNCH

In the case of a symmetrical wedge-shaped punch $g_1^\pm=\pm g_1, g_1\in(0,1)$ and for $S_{1,2}(\rho)$ for the following expressions hold

$$S_1(\rho)=T\sin(\tau\rho+w(\rho)),\quad S_2(\rho)=T\cos(\tau\rho+w(\rho)) \tag{4.1}$$

$$T\equiv\sqrt{a_*^2+b_*^2}=\varkappa\{[(D_1-\Lambda_1)^2+\Lambda_2^2][(B_0+g_1\Lambda_1)^2+g_1^2\Lambda_2^2]\}^{1/2}$$

$$a_*=\varkappa[g_1\Lambda_2^2+(D_1-\Lambda_1)(B_0+g_1\Lambda_1)]$$

$$b_*=\varkappa[(D_1-\Lambda_1)g_1-(B_0+g_1\Lambda_1)]\Lambda_2$$

$$\sin w=b_*/T,\quad \cos w=a_*/T$$

$$B_0=\delta_0-g_1\gamma_0,\quad D_1=\gamma_0-\mu g_1\delta_0,\quad \varkappa=8g_1\delta_0\cos\omega/\operatorname{ch}\rho$$

Henceforth we will eliminate from consideration the case when $D_1=0$, which by virtue of the first relation of (3.5), only occurs when $\mu=1$ and $g_1=\gamma_0/\delta_0$.

It follows directly from the expression for T given above, in which $D_1\neq 0, g_1>0$ and, according to (3.5), $\Lambda_2\neq 0$ when $\rho\neq 0$, that $T>0$. The last inequality enables us, using (4.1), to give Eq. (3.8) for ρ the following form

$$n=\operatorname{tg}(\tau\rho+w(\rho)) \tag{4.2}$$

Moreover, it can be established that for values of ρ which satisfy Eq. (4.2), the third equation of system (3.4) is equivalent to the first and hence should be eliminated from consideration. The remaining first two equations of (3.4) give

$$\varphi_1^-=d_1/d_0,\quad \varphi_1^+=d_2/d_0\quad \text{for}\ d_0\neq 0 \tag{4.3}$$

Hence, for a symmetrical wedge-shaped punch, after ρ has been determined from Eq. (4.2), the parameters $\varphi_1^\pm$ of the function $\varphi(x)$ can be found from Eqs (4.3) provided $d_0\neq 0$. Expressions for the stresses $q_{1,2}(x,a)$ are found directly from Eqs (3.11).

Substituting (4.3) into (3.12) and using (3.9) we obtain the following expression for $P_2(a)$

$$P_2(a)=\tfrac{1}{2}m_0(1+e^{2\rho})S_2d_0^{-1}a \tag{4.4}$$

When (4.2), (4.3) and (4.4) are satisfied, we can give inequalities (3.13) the form

$$1+d_k/d_0>0,\quad k=1,2;\quad S_2/d_0>0 \tag{4.5}$$

In conclusion we will consider the case of small values of $|\rho|$. For this purpose we will write the following asymptotic forms of the functions $\Lambda_{1,2}(\rho)$

$$\Lambda_1(\rho)=\tfrac{1}{2}\tau\rho^2+O(\rho^4),\quad \Lambda_2(\rho)=\rho+O(\rho^3)$$

by means of which we obtain from (3.10)

$$d_0(\rho) = 2D_1D_2 + O(\rho^2), \quad d_{1,2}(\rho) = -2g_1\gamma_0 D_1 \mp 2g_1\delta_0\rho + O(\rho^2) \tag{4.6}$$
$$B = B_0 \cos\omega, \quad D_2 = \delta_0 + \mu g_1\gamma_0$$

and we represent Eq. (4.2) for ρ in the form

$$n = n_1\rho + O(\rho^3), \quad n_1 = [(g_1 \cos\omega + \tau B)D_1 - B](BD_1)^{-1} \tag{4.7}$$

Note that the parameters B and D_2, which occur in (4.6) and (4.7), are non-zero—this follows from inequalities (3.5) and the constraint $g_1 \in (0, 1)$. Moreover, it can be established that $n_1 \neq 0$ and we can obtain from (4.7)

$$\rho = n/n_1 + \varphi(n^3) \tag{4.8}$$

Substituting (4.6) and (4.8) into (3.6), (4.3) and (4.4) we obtain

$$b(a) = (1 + 2n/n_1)a + O(n^2)$$
$$\varphi_1^{\pm} = g_1 D_1^{-1}[-\gamma_0 \pm \delta_0(D_1n_1)^{-1}n] + O(n^2) \tag{4.9}$$
$$P_2(a) = 4g_1\delta_0 BD_2^{-1}m_0(1 + n/n_1)a + O(n^2)$$

Hence, as $\rho \to 0$ we have $n \to 0$ and expression (4.9) is the solution of the problem. The stresses $q_{1,2}(x, a)$ are found from (3.11) using the values of $\varphi_1^{\pm}$ obtained. Verification of conditions (4.5) presents no difficulty.

When $n = 0$, expressions (4.9) and expressions (3.11) for the contact stresses, corresponding to them, give the solution of the symmetrical problem with adhesion for a wedge-shaped punch, considered previously [1].

REFERENCES

1. SOLDATENKOV, I. A., Indentation with adhesion of a symmetrical punch into an elastic half-plane. *Prikl. Mat. Mekh.*, 1996, **60**, 267–273.
2. GOODMAN, L. E., Contact stress analysis of normally loaded rough spheres. *Trans. ASME. Ser. E. J. Appl. Mech.*, 1962, **29**, 515–522.
3. MOSSAKOVSKII, V. I., Compression of elastic solids under adhesion conditions (the axisymmetric case). *Prikl. Mat. Mekh.*, 1963, **27**, 418–427
4. MOSSAKOVSKII, V. I. and FOTIYEVA, N. N., Indentation of a symmetrical punch into an elastic half-plane when there is adhesion along the linear of contact. *Izv. Akad. Nauk SSSR. Mekhanika*, 1965, 6, 67–70.
5. MUSKHELISHVILI, N. I., *Singular Integral Equations. Boundary-value Problems of the Theory of Functions and Some of their Applications to Mathematical Physics*. Nauka, Moscow, 1968.
6. SOLDATENKOV, I. A., The contact problem for a half-plane taking the tangential displacement at the contact into account. *Izv. Ross. Akad. Nauk. MTT*, 1994, 4, 51–61.
7. GALIN, L. A., *Contact Problems of the Theory of Elasticity and Viscoelasticity*. Nauka, Moscow, 1980.
8. BEKLEMISHEV, D. V., *A Course in Analytical Geometry and Linear Algebra*. Nauka, Moscow, 1974.

Translated by R.C.G.